Ian Mitroff 1/13/2009

Visiting Professor, Haas School of Business/IBER, and Senior Investigator, CCRM

Preliminary Models on the **Probability** of Implication

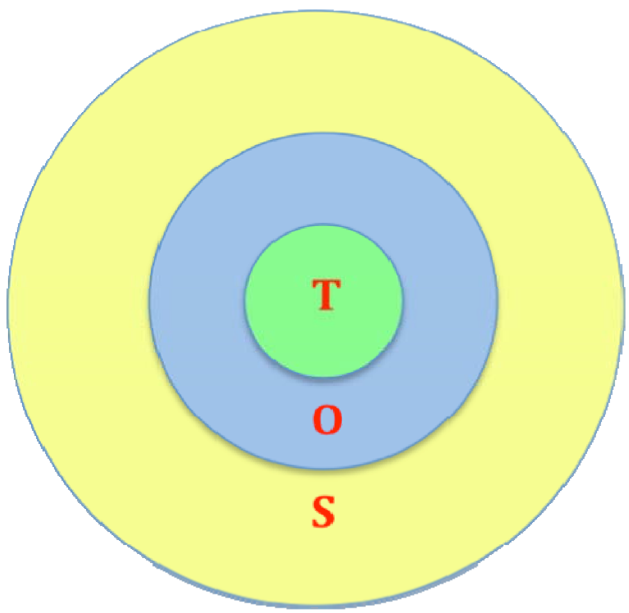


Figure 1

All technologies (T) are embedded within an organization (O), or a set of organizations, which in turn are embedded within a society (S) (Figure 1 above). Indeed, political scientists and sociologists contend that society is the both sum and the product of the interactions among its institutions and organizations.

From the failure of O's, we are trying to infer the linkage(s) between O's and T's; that is, we are trying to infer the relationship: the failure of O leads to the failure of T, or O leads to T. We are also trying to deduce the failure of T's; that is, given O and O leads to T, therefore T; and vice versa: given T and T leads to O, therefore O. Even more, we are trying to infer P[O], the probability of the failure of O, and P[O leads to T], i.e., the probability of the failure of O leads to the failure of T. From these, we are trying to deduce P[O and O leads to T], and therefore, finally, P[T].

1. Background: Logical Implication

Modus ponens: $[O \& (O \rightarrow T)] \rightarrow T$, where "$\rightarrow$" is defined as "implies." (Modus ponens is one of the first "laws of logic" that the Greeks discovered/invented several millennia ago.)

For brevity, "is defined as" will be represented by "def." "Is identical to" will be represented by "idt."

$(O \rightarrow T)$ idt [not (O & not-T)] because it cannot be the case that the antecedent of an argument, O, is true and the consequent, T, is false.

Note that if "O def the failure of an O," then "not-O def is the non failure of an O; in other words, not-O is equivalent to the continued "successful operation" of an O. T def the failure of T.

(O → T) def "**If** O fails, **then** T fails." Thus, "→" is the "if, then" part of an argument. It may also be thought of as the "logical or conceptual linkage between O and T."

(O → T) idt [not (O & not-T)] idt [not-O or T] because of the identities: not (a & b) idt [(not- a) or (not-b)] and [not (a or b)] idt [(not-a) & (not-b)].

Therefore, modus ponens is true (a correct form of reasoning) because:

{[O & (O → T)] → T} idt {not [O & (not-O & T)] or T }

idt not-O or not (not-O or T) or T

idt (not-O or T) or not-(not-O or T)

idt (p or not-p) idt Logical Truth def LT. (This follows because of the def of the logical operator "or" and "not-p" via a simple truth-table.

Model 1A: A Simple Model of Sufficiency

(a) For simplicity, let $P[O \,\&\, O \rightarrow T] = P[T]$, where $P[O \rightarrow T]$ is the probability that the implication $O \rightarrow T$ "applies" or "holds." Loosely, it can also be thought of as the "strength" of the implication, or linkage.

"Sufficiency" means that O is sufficient for T. Given $(O \rightarrow T)$, then the occurrence of O is sufficient for the occurrence of T. In other words, since we defined O as "the failure of an O," if O fails, then T fails as well.

Note: Bob Bea's Qmas methodology derives values for P[O], P[T], and $P[O \rightarrow T]$. Hence, there is a linkage between this effort here and Bob's work.

$P[O \,\&\, O \rightarrow T]= P[O/O \rightarrow T]\, P[O \rightarrow T]= P[O \rightarrow T/O]\, P[O]$.

If we let $P[O \rightarrow T/O]= P[T]$, then we are saying that given O, i.e., the occurrence of O, $P[O \rightarrow T/O]$ is equivalent to $O \,\&\, O \rightarrow T$, which implies T.

$P[O \rightarrow T/O]\, P[O]= P[T]\, P[O]=P[T]$ from (a).

(b) Therefore, P[O]=1.

$P[O/O \rightarrow T]\, P[O \rightarrow T]=P[T]$

If we let $P[O/O \rightarrow T]=P[O]$, then we are saying that O is not conditioned by $O \rightarrow T$.

$P[O]\ P[O \rightarrow T]=P[T]$. Therefore, from (b),

(c) $P[O \rightarrow T]=P[T]$.

Notice also that $P[O \rightarrow T]=P[\text{not-}O \text{ or } T]= P[\text{not-}O]+P[T] -$

$P[\text{not-}O \ \& \ T]$.

If $P[\text{not-}O \ \& \ T]=0$ because T cannot fail without O failing, then

$P[O \rightarrow T]=1-P[O]+P[T]=P[T]$ because of (b).

Therefore, again, $P[O \rightarrow T]=P[T]$.

(d) $0<= P[O \rightarrow T]=P[T]<=P[O]=1$.

<u>Model 1B: Sufficiency</u>

If we let $P[O/O \rightarrow T]=P[T]$, then

$P[O/O \rightarrow T]\ P[O \rightarrow T]=P[T]\ P[O \rightarrow T]=P[T]$.

(e) Therefore, $P[O \rightarrow T]=P[T]=1$.

(f) P[O → T]=P[T]=P[O]=1.

Model 1C: Necessity

Necessity means that if O does *not* occur, then T will *not* occur as well. That is, if O does not fail, then T will not fail as well. Since O and T were initially defined as failure, the not-O and not-T mean the successful operation of O and T. Thus, necessity def (not-O → not-T).

By simple substitution of not-O for O, and not-T for T in (d), we get:

(g) 0<= P[not-O → not-T]=P[not-T]<=P[not-O]=1.

Model 1D: Necessity

(h) P[not-O → not-T]=P[not-T]=P[not-O]=1.

Substituting {1-P[O]} for P[not-O] and {1-P[T]} for P[not-T], we get

(i) P[T]=P[O]=P[T → O]=0 for (g).

(j) $P[T]=P[O]<P[T \rightarrow O]=1$ for (h).

Note that (not-O $\rightarrow$ not-T) idt (O or not-T) idt (T$\rightarrow$O).

But this means that for the simple Model 1 outlined above, it is not possible to get both sufficiency and necessity simultaneously. By itself, this is enough, i.e., "sufficient," to eliminate it and not to explore it further.

Model 2: Necessity and Sufficiency

P[O $\rightarrow$ T]=P[not-O & T]=P[not-O]+P[T]-P[not-O & T].

But, (not-O & T) idt not(O or not-T).

Therefore, P[O $\rightarrow$ T]={1-P[O]}+P[T]-{1-P[O or not-T]}.

P[O $\rightarrow$ T]={1-P[O]}+P[T]-{1-P[O or not-T]}.

P[O $\rightarrow$ T]=1-P[O]+P[T]-1+P[O or not-T].

P[O $\rightarrow$ T]=1-P[O]+P[T]-1+P[T$\rightarrow$O].

Therefore,

(k) $\Delta=P[O \rightarrow T]- P[T\rightarrow O]=P[T]-P[O]$

Or, *the probability of sufficiency <u>minus</u> the probability of necessity <u>equal</u>s the probability that T fails <u>minus</u> the probability that O fails. In other words,* $P[O \rightarrow T]$ *&* $P[T \rightarrow O]$ *do* ***not*** *necessarily imply one another.* $P[O \rightarrow T] = P[T \rightarrow O]$ *only for the special case where* $P[O] = P[T]$*. In general,* $P[O \rightarrow T] <> P[T \rightarrow O]$*. See Figure 2 below.*

In order to explore Model 2 further, consider two special cases: (1) $P[T] = - P[O] + k$; and, (2) $P[T] = + P[O] + k$.

(1) $P[T] = - P[O] + k$; therefore, $\Delta = -2\,P[O] + k$.

$$P[O \rightarrow T] = P[T \rightarrow O] - 2\,P[O] + k, \text{ and conversely,}$$

$$P[T \rightarrow O] = P[O \rightarrow T] + 2\,P[O] - k.$$

If $P[T] = P[O \;\&\; O \rightarrow T] = P[O]\,P[O \rightarrow T] = - P[O] + k$, then

(l) $P[O \rightarrow T] = (-P[O] + k) / P[O]$.

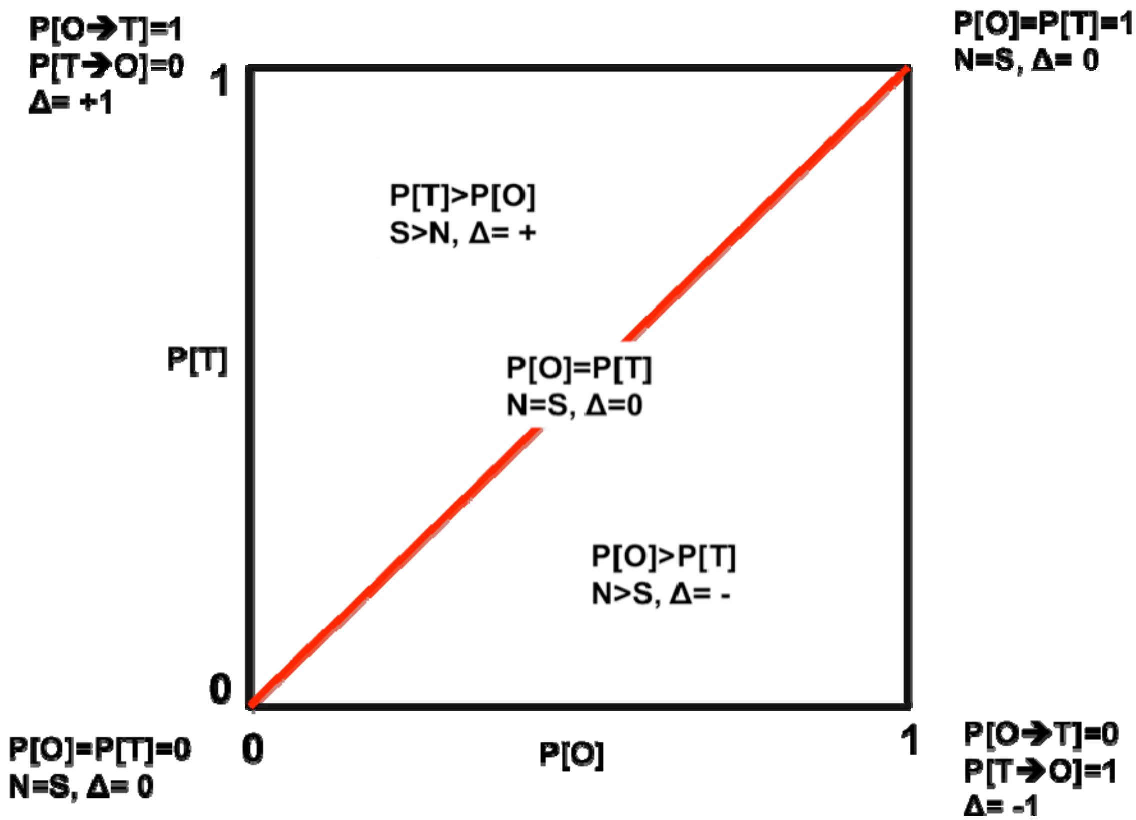


Figure 2: P[T]=P[O] + Δ

(2) P[T] = + P[O] + k; therefore, **Δ=** k.

P[O → T] = P[T→O] + k, and conversely,

P[T → O] = P[O→T] – k.

If P[T] = P[O & O→ T] = P[O] P[O→ T] = - P[O] +k, then

(m) P[O→ T] = (P[O] +k)/ P[O].

To explore the model further, consider the case where

C = constant = P[O→ T] = (P[O] +k)/ P[O]. Then, it follows that

(n) (C-1) P[O] = k, or

(o) P[O] = k / (C-1).

Since P[T] = P[O] + k, it also follows that

(p) P[T] = k C / (1-C). Therefore,

(q) P[T] = C P[O]. From these relationships and (k) above, it can also be shown that

(r) P[T→O] = (1-C) P[O] + C, where 0=< C <= 1.

Results for (q) and (r) for various values of C are given in Figure 3 below

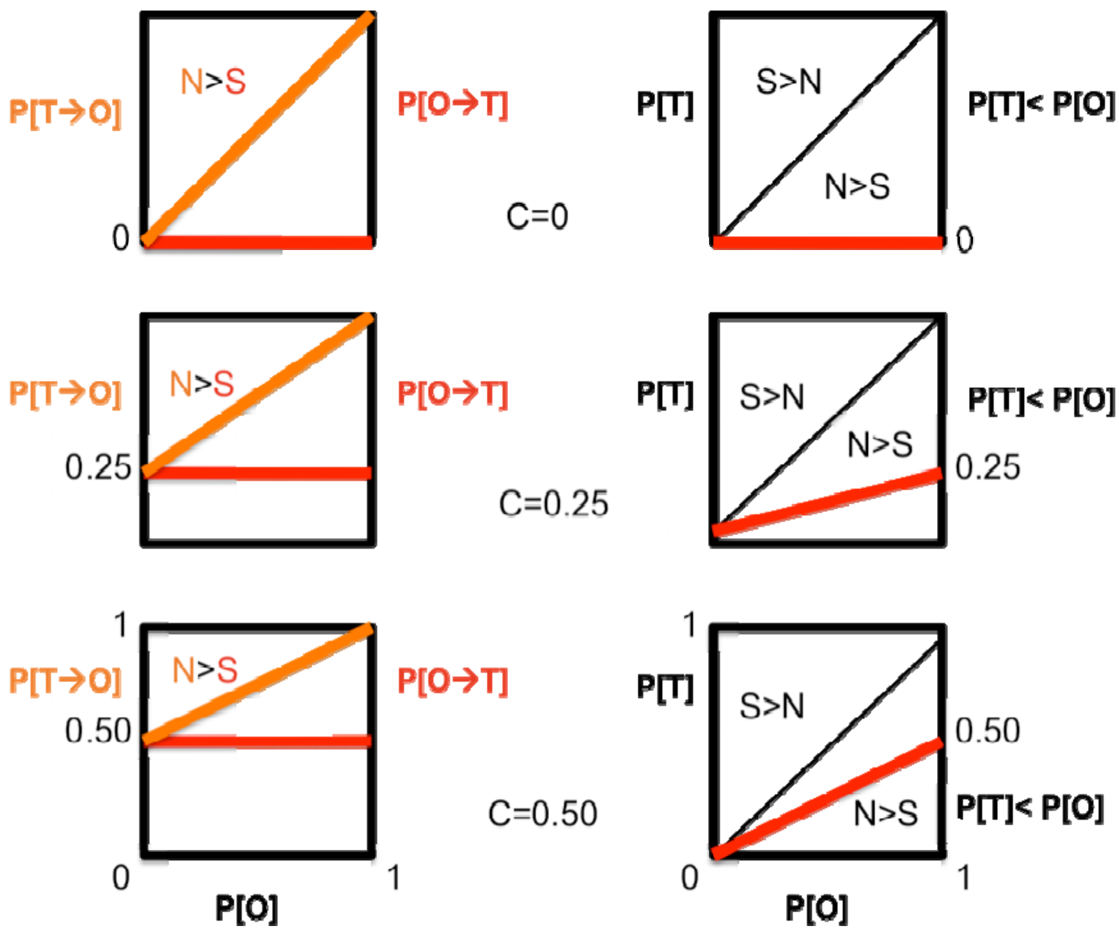

N>S
P[T→O]
P[O→T]
0
C=0
P[T]
S>N
N>S
P[T]< P[O]
0
N>S
P[T→O]
P[O→T]
0.25
C=0.25
P[T]
S>N
N>S
P[T]< P[O]
0.25
1
N>S
P[T→O]
P[O→T]
0.50
C=0.50
0
1
P[O]
1
P[T]
S>N
0.50
N>S
P[T]< P[O]
0
1
P[O]

Figure 3

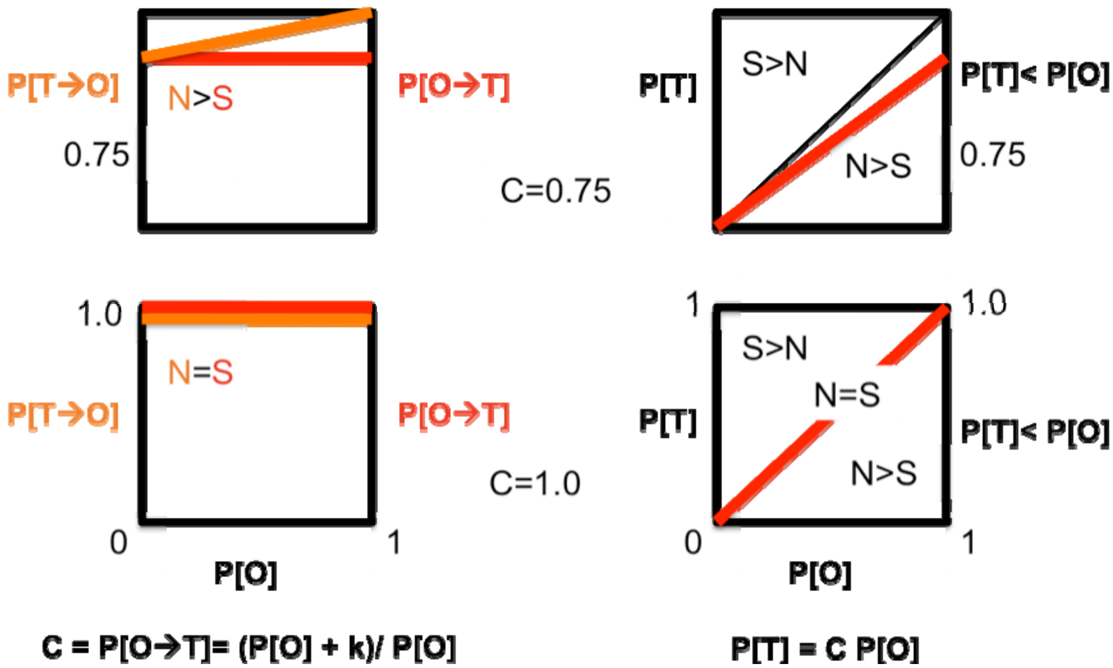
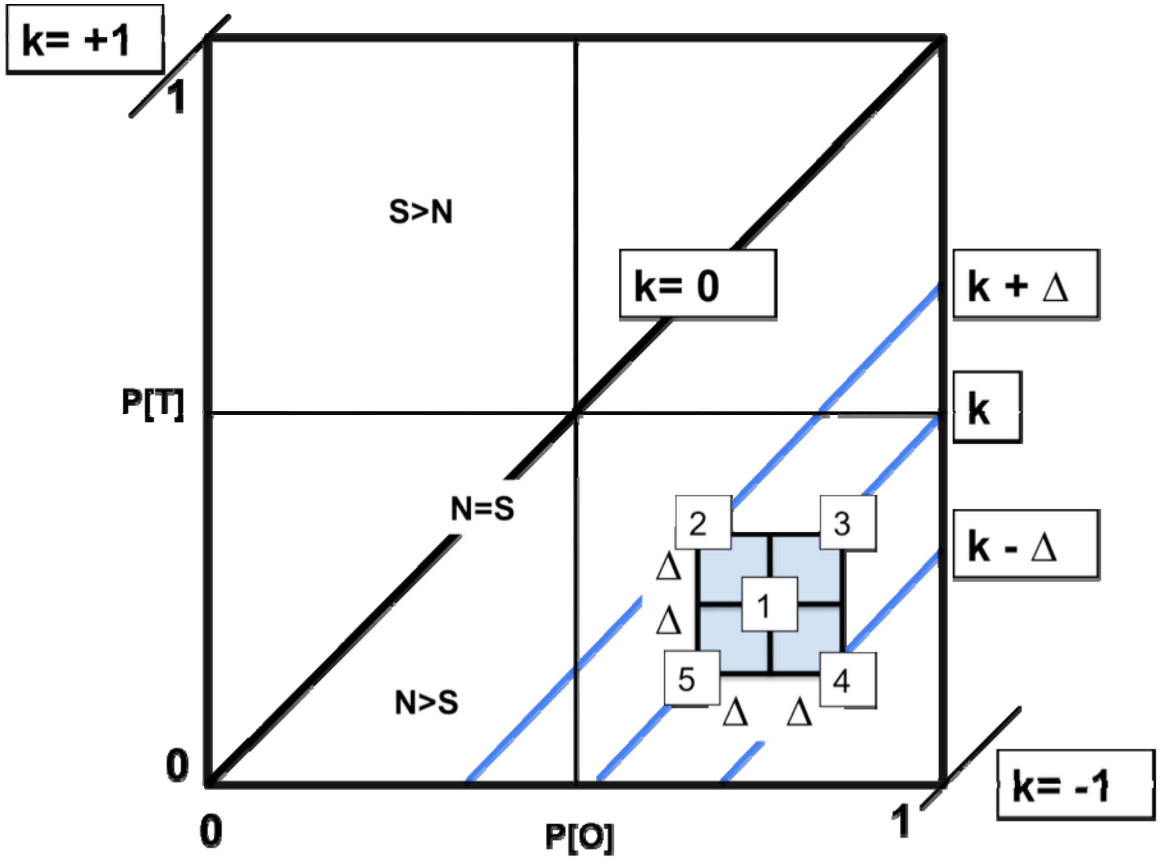


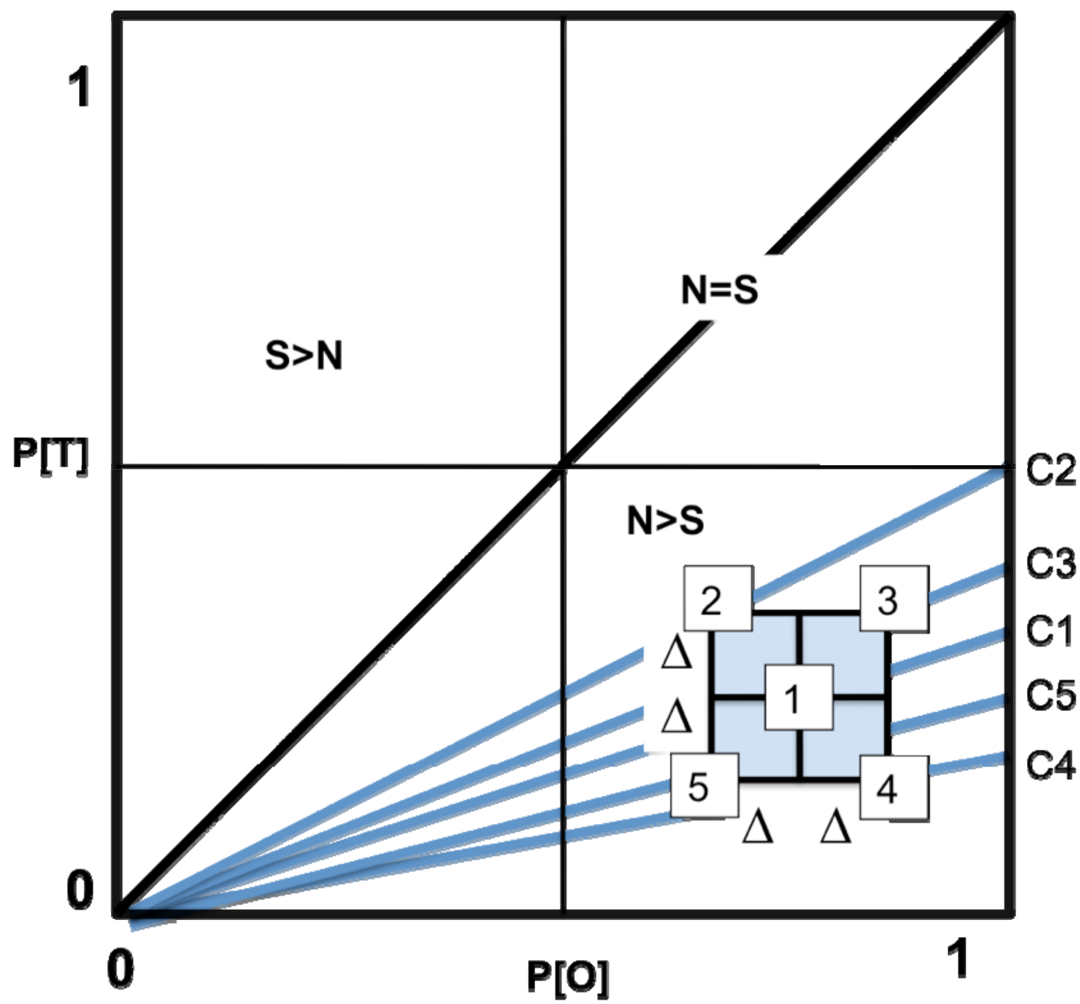
Figure 3 (continued)

Finally, Figure 3 can also be used to define regions of interest as follows:



(1) P[T] = P[O] + k

	k	P[O]	P[T] = P[O] + k	P[O→T] = P[T] / P[O]
1	k	P[O]	P[O] + k	P[T] / P[O]
3	k	P[O] +Δ	[P[O] +k] + Δ	{[P[O] +k] + Δ} / P[O] + Δ
5	k	P[O] - Λ	[P[O] +k] - Δ	{[P[O] +k] - Δ} / P[O] - Δ
2	k + Δ	P[O] - Δ	[P[O] +k] + Δ	{[P[O] +k] + Δ} / P[O] - Δ
4	k - Δ	P[O] + Δ	[P[O] +k] - Δ	{[P[O] +k] - Δ} / P[O] + Δ



$P[T] = C \;\; P[O]$

The upshot is that what we can conclude from P[O], P[O & O→T], and P[O→T] is strongly dependent on the assumptions (models) that we are willing to postulate, i.e., the probabilities of the logical relationships between O, (O & O→T), and (O→T).

Preliminary Models on the **Plausibility** of Implication

In a paper ("Policy As Argument—A Logic for Ill-Structured Decision Problems," Management Science, Vol. 28, No. 12, December, 1982, pp. 1391-1404), Richard Mason, Vince Barabba, and I developed an alternate approach that is based on ***plausibilities***, not probabilities. The difference is as follows: ***probabilities* refer to events; *plausibilities* to arguments.** For instance, an argument for some assertion or proposition can be highly plausible even if the events that are part of the assertion or proposition are highly improbable or unlikely such as 9/11--until of course the event actually occurs. In other words, an argument is plausible if it is coherent, makes sense, and it is well structured. (Classic examples are found in the philosophy of religion; many, if not most, of the philosophical arguments for the existence of God are quite plausible even if they are improbable. But then so are the arguments for the non-existence of God as well.)

1. Background: Plausibility Indexing and Ranking

From elementary logic, it can be shown that:

(a & b)→ a→ (a or b) , and (a & b)→ b→ (a or b).

Therefore, pl(a & b) <= pl(a) <= pl(a or b), and

pl (a & b) <= pl(a) <= pl(a or b), where pl def "the plausibility of."

The above follows because:

pl(a & not-a) <= pl(a) <= pl(a or not-a).

In fact, pl(a & not-a) = 0 because (a & not-a) is a logically false statement or proposition , and (a or not-a) is a logically true statement or proposition. (For example, either it is raining or it is not raining.)

Therefore, pl(a or not-a) has a maximal pl index or ranking,

and pl (a & not-a) has a minimal pl index or ranking.

Thus, we can (arbitrarily) set pl(a or not-a) = 10,

and pl (a & not-a) = 0 as the two anchor points of the scale.

Furthermore, if (a_1 & a_2 & …ai &…&a_n)→a_{n+1}, then

pl(a_{n+1}) = min pl (a_i); in other words, the plausibility of the consequent of an argument can not be greater than the weakest link of the chain of the entire argument.

2. <u>O & O→T & T as Parts of an Argument Structure</u>

pl(O & O→ T) = pl(T); therefore, pl(T)= min (pl(O), pl(O→ T).

Furthermore, since (O→ T) = (not-O or T), then

pl(O→ T) = pl(not-O or T).

And, pl(not-O or T) => pl(not-O), and

pl(not-O or T) => pl(T).

We thus have the following:

(a) pl(T) = pl(O) <= pl(O→T), or

(b) pl(T) = pl(O→T) <= pl(O).

We also have:

(c) pl(O→T) = pl(not-O or T) => pl(not-O) => pl(T), or

(d) pl(O→T) = pl(not-O or T) => pl(T) => pl(not-O).

In addition, we are interested in pl(O&T) because O&T def as the failure of O and T. That is, we are not just interested in the failure of O and T alone, but in the joint failure of O and T .

Since pl(O&T) = min [pl(O), pl(T)], we also have:

(e) pl(O&T) = pl(O) <= pl(T).

(f) pl(O&T)] = pl(T) <= pl(O).

Putting (a), (b), (c), (d), (e), and (f) together in all possible combinations, we obtain:

(1) $pl(O \rightarrow T) = pl(T) = pl(O) = pl(O\&T) = pl(not\text{-}O)$.

(2) $pl(O \rightarrow T) >= pl(T) = pl(O) = pl(O\&T) >= pl(not\text{-}O)$.

(3) $pl(O \rightarrow T) >= pl\ [T] >= pl(not\text{-}O) >= pl(O) = pl(O\&T)$.

(4) $pl(O) >= pl(T) = pl(O \rightarrow T) = pl(O\&T) = pl(not\text{-}O)$.

From plausibility indexing, it can also be shown that:

(g) $pl(O\&T) = [\ pl(O) + pl(O \rightarrow T) + pl(T)\ ] / 3$.

To avoid confusion, we shall call pl(O&T) in (g), $pl(O\&T)_{AV}$

And pl(O&T) in (1), (2), …(4), $pl(O\&T)_{Logic}$. Putting $pl(O\&T)_{AV}$ in (1), (2), …(4) results in:

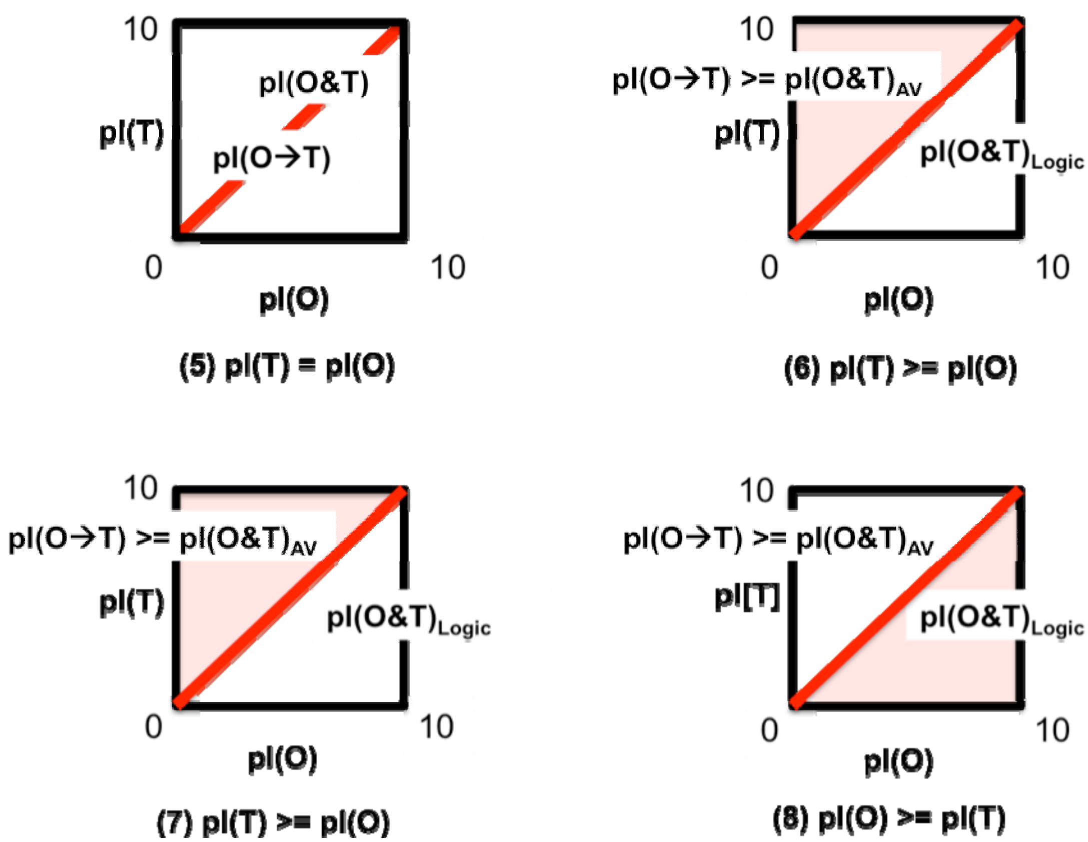
(5) $pl(O \rightarrow T) = pl(T) = pl(O) = pl(O\&T)_{Logic} = pl(not\text{-}O) = pl(O\&T)_{AV.}$

(6) $pl(O \rightarrow T) >= pl(O\&T)_{AV} = pl(T) = pl(O) = pl(O\&T)_{Logic} >= pl(not\text{-}O)$.

(7) pl(O→T) >= $pl(O\&T)_{AV}$ >= pl [T] >= pl(not-O) >= pl(O) = $pl(O\&T)_{Logic}$.

(8) pl(O) >= $pl(O\&T)_{AV}$ >= pl(T) = pl(O→T) = $pl(O\&T)_{Logic}$ pl(not-O).

The Figure below shows the cases (5) through (8).



The end result is that depending upon the assumptions we are willing to make, there is considerable latitude in the assignment***s*** of plausibilities. In spite of this, the assignments are not arbitrary. It cannot be emphasized too strongly that they reflect both what we presume to know and we feel justified in assuming. In other words, how plausible we feel our assumptions are.

Notice also that if we can assign a probability function to cases (5) through (8), then we have another way to compute the Type Three Error or E3. In this situation, E3 becomes P[pl], i.e., the probability of a particular plausibility!